

Generalized Induced Norms*

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Abstract

Let $\|\cdot\|$ be a norm on the algebra M_n of all $n \times n$ matrices over \mathbb{C} . An interesting problem in matrix theory is that "are there two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on \mathbb{C}^n such that $\|A\| = \max\{\|Ax\|_2 : \|x\|_1 = 1\}$ for all $A \in M_n$. We will investigate this problem and its various aspects and will discuss under which conditions $\|\cdot\|_1 = \|\cdot\|_2$.

1 Preliminaries

Throughout the paper M_n denotes the complex algebra of all $n \times n$ matrices $A = [a_{ij}]$ with entries in \mathbb{C} together with the usual matrix operations. Denote by $\{e_1, e_2, \dots, e_n\}$ the standard basis for \mathbb{C}^n , where e_i has 1 as its i th entry and 0 elsewhere. We denote by E_{ij} the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere.

For $1 \leq p \leq \infty$ the norm ℓ_p on \mathbb{C}^n is defined as follows:

$$\ell_p(x) = \ell_p\left(\sum_{i=1}^n x_i e_i\right) = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} & 1 \leq p < \infty \\ \max\{|x_1|, \dots, |x_n|\} & p = \infty \end{cases}$$

A norm $\|\cdot\|$ on \mathbb{C}^n is said to be unitarily invariant if $\|x\| = \|Ux\|$ for all unitaries U and all $x \in \mathbb{C}^n$.

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By an algebra norm (or a matrix norm) we mean a norm $\|\cdot\|$ on M_n such that $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in M_n$. An algebra norm $\|\cdot\|$ on M_n is called unitarily invariant if $\|UAV\| = \|A\|$ for all unitaries U and V and all $A \in M_n$. See [2, Chapter IV] for more information.

Example 1.1 The norm $\|A\|_\sigma = \sum_{i,j=1}^n |a_{ij}|$ is an algebra norm, but the norm $\|A\|_m = \max\{|a_{i,j}| : 1 \leq i, j \leq n\}$ is not an algebra norm, since $\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|_m^2 > \left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|_m^2$.

Remark 1.2 It is easy to show that for each norm $\|\cdot\|$ on M_n , the scaled norm $\max\{\frac{\|AB\|}{\|A\|\|B\|} : A, B \neq 0\} \|\cdot\|$ is an algebra norm; cf. [1, p.114]

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on \mathbb{C}^n . Then for each $A : (\mathbb{C}^n, \|\cdot\|_1) \rightarrow (\mathbb{C}^n, \|\cdot\|_2)$ we can define $\|A\| = \max\{\|Ax\|_2 : \|x\|_1 = 1\}$. If $\|\cdot\|_1 = \|\cdot\|_2$, then $\|I\| = 1$ and there are many examples of $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $\|I\| \neq 1$. This shows that given $\|\cdot\|$ on M_n , we cannot deduce in general that there is a norm $\|\cdot\|_1$ on \mathbb{C}^n with $\|A\| = \max\{\|Ax\|_1 : \|x\|_1 = 1\}$. Let us recall the concept of g-ind norm as follows:

Definition 1.3 Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on \mathbb{C}^n . Then the norm $\|\cdot\|_{1,2}$ on M_n defined by $\|A\|_{1,2} = \max\{\|Ax\|_2 : \|x\|_1 = 1\}$ is called the generalized induced (or g-ind) norm via $\|\cdot\|_1$ and $\|\cdot\|_2$. If $\|\cdot\|_1 = \|\cdot\|_2$, then $\|\cdot\|_{1,1}$ is called induced norm.

Example 1.4 $\|A\|_C = \max\{\sum_{i=1}^n |a_{i,j}| : 1 \leq j \leq n\}$, $\|A\|_R = \max\{\sum_{j=1}^n |a_{i,j}| : 1 \leq i \leq n\}$ and the spectral norm $\|A\|_S = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\}$ are induced by ℓ_1, ℓ_∞ and ℓ_2 (or the Eucledian norm), respectively.

It is known that the algebra norm $\|A\| = \max\{\|A\|_C, \|A\|_R\}$ is not induced [] and it is not hard to show that it is not g-ind too; cf. [1, Corollary 3.2.6]

We need the following proposition which is a special case of a finite dimensional version of the Hahn-Banach theorem [5, p. 104]:

Proposition 1.5 *Let $\|\cdot\|$ be a norm on \mathbb{C}^n and $y \in \mathbb{C}^n$ be a given vector. There exists a vector $y_\circ \in \mathbb{C}^n$ such that $y_\circ^* y = \|y\|$ and for all $x \in \mathbb{C}^n$, $|y_\circ^* x| \leq \|x\|$. (Throughout, $*$ denotes the transpose) [3, Corollary 5.5.15])*

In this paper we examine the following nice problems:

- (i) Given a norm $\|\cdot\|$ on M_n is there any class \mathcal{A} of M_n such that the restriction of the norm $\|\cdot\|$ on \mathcal{A} is g-ind?
- (ii) When a g-ind norm is unitarily invariant?
- (iii) If a given norm $\|\cdot\|$ is g-ind via $\|\cdot\|_1$ and $\|\cdot\|_2$, then is it possible to find $\|\cdot\|_1$ and $\|\cdot\|_2$ explicitly in terms of $\|\cdot\|$?
- (iv) When two g-ind norms are the same?
- (v) Is there any characterization of the g-ind norms which are algebra norms?

2 Main Results

We begin with some observations on generalized induced norms.

Let $\|\cdot\|_{1,2}$ be a generalized induced norm on M_n obtained via $\|\cdot\|_1$ and $\|\cdot\|_2$. Then $\|E_{ij}\|_{1,2} = \max\{\|E_{ij}x\|_2 : \|x\|_1 = 1\} = \max\{\|x_j e_i\|_2 : \|(x_1, \dots, x_j, \dots, x_n)\|_1 = 1\} = \alpha_j \|e_i\|_2$, where $\alpha_j = \max\{|x_j| : \|(x_1, \dots, x_j, \dots, x_n)\|_1 = 1\}$. In general, for $x \in \mathbb{C}^n$ and $1 \leq j \leq n$, if $C_{x,j} \in M_n$ is defined by the operator $C_{x,j}(y) = y_j x$ then $\|C_{x,j}\|_{1,2} = \alpha_j \|x\|_2$.

Also if for $x \in \mathbb{C}^n$ we define $C_x \in M_n$ by $C_x = \sum_{j=1}^n C_{x,j}$, then clearly $\|C_x\|_{1,2} = \alpha \|x\|_2$, where $\alpha = \max\{|\sum_{j=1}^n y_j| : \|(y_1, \dots, y_j, \dots, y_n)\|_1 = 1\}$.

Now we give a partial solution to Problem (i) and useful direction toward solving Problem (iii):

Proposition 2.1 *Let $\|\cdot\|$ be an algebra norm on M_n . Then $\|\cdot\|$ is a g-ind norm on $\{A \in M_n : \|A\| = \|A^{-1}\| = 1\}$.*

Proof. Put $\|x\|_1 = \max\{\|C_{Ax}\| : \|A\| = 1\}$, $\lambda^{-1} = \max\{|\sum_{i=1}^n x_i| : \|x\|_1 = 1\}$ and $\|x\|_2 = \lambda \|C_x\|$.

Then we have $\|C_y\|_{1,2} = \max\{\|C_y x\|_2 : \|x\|_1 = 1\} = \max\{|\sum_{i=1}^n x_i| \|y\|_2 : \|x\|_1 = 1\} = \|y\|_2 \lambda^{-1} = \|C_y\|$.

It follows that for each $y \in \mathbb{C}^n$ there is some $x \in \mathbb{C}^n$ such that $\|C_y x\|_2 = \|C_y\| \|x\|_1 = \|C_y\| \max\{\|C_{Dx}\| : \|D\| = 1\}$.

Now let A be invertible and $\|A^{-1}\| = \|A\| = 1$ and $z = A^{-1}C_y x$. Then $\lambda^{-1}\|Bz\|_2 = \lambda^{-1}\|BA^{-1}C_y x\|_2 = \lambda^{-1}\|Dx\|_2 = \|C_{Dx}\| \leq \frac{1}{\|C_y\|} \|C_y x\|_2 = \frac{1}{\|C_y\|} \|Az\|_2$.

Now choose y so that $\|C_y\| = 1$. Then $\|C_{Bz}\| \leq \|C_{Az}\|$ for all $B \in M_n$. This implies that $\|C_{Az}\|$ is an upper bound for the set $\{\|C_{Bz}\| : \|B\| = 1\}$ and indeed $\|C_{Az}\| = \max\{\|C_{Bz}\| : \|B\| = 1\} = \|z\|_1$. It follows that $\|A\| = 1 = \|C_{A(\frac{z}{\|z\|_1})}\| = \max\{\|C_{Au}\| : \|u\|_1 = 1\} = \max\{\|Au\|_2 : \|u\|_1 = 1\} = \|A\|_{1,2}$. \square

Let us now answer Question (ii).

Proposition 2.2 *An induced norm $\|\cdot\|_{1,2}$ is unitarily invariant if and only if so are $\|\cdot\|_1$ and $\|\cdot\|_2$.*

Proof. Let U, V be unital operators and A be an arbitrary operator on \mathbb{C}^n .

Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are unitarily invariant. Then

$$\|UAV\|_{1,2} = \max_{x \neq 0} \frac{\|UAVx\|_2}{\|x\|_1} = \max_{x \neq 0} \frac{\|AVx\|_2}{\|x\|_1} = \max_{y \neq 0} \frac{\|Ay\|_2}{\|V^{-1}x\|_1} = \max_{y \neq 0} \frac{\|Ay\|_2}{\|y\|_1} = \|A\|_{1,2}.$$

Conversely, if $\|\cdot\|_{1,2}$ is unitarily invariant, then $\|Ux\|_1 = \max\{\|AUx\|_2 : \|A\|_{1,2} \leq 1\} = \max\{\|Bx\|_2 : \|U^{-1}B\|_{1,2} \leq 1\} = \max\{\|Bx\|_2 : \|B\|_{1,2} \leq 1\} = \|x\|_1$ and $\|Ux\|_2 = \frac{1}{\alpha} \|C_{Ux}\| = \frac{1}{\alpha} \|UC_x\| = \frac{1}{\alpha} \|C_x\| = \|x\|_2$. \square

Modifying the proof of Theorem 5.6.18 of [3], we obtain a similar useful result for g-ind norms:

Theorem 2.3 *Let $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_3$ and $\|\cdot\|_4$ be four given norms on \mathbb{C}^n and*

$$R_{i,j} = \max\left\{\frac{\|x\|_i}{\|x\|_j} : x \neq 0\right\}, 1 \leq i, j \leq 4.$$

Then

$$\max\left\{\frac{\|A\|_{1,2}}{\|A\|_{3,4}} : A \neq 0\right\} = R_{2,4} R_{3,1}$$

In particular, $\max\{\frac{\|A\|_{1,1}}{\|A\|_{2,2}} : A \neq 0\} = \max\{\frac{\|A\|_{2,2}}{\|A\|_{1,1}} : A \neq 0\} = R_{1,2}R_{2,1}$.

Proof. Let A be a matrix and $x \neq 0$. Then $\frac{\|Ax\|_2}{\|x\|_1} = \frac{\|Ax\|_2}{\|Ax\|_4} \cdot \frac{\|Ax\|_4}{\|x\|_3} \cdot \frac{\|x\|_3}{\|x\|_1}$. Hence $\|A\|_{1,2} \leq R_{2,4}\|A\|_{3,4}R_{3,1}$. Thus $\frac{\|A\|_{1,2}}{\|A\|_{3,4}} \leq R_{2,4}R_{3,1}$.

There are vectors y, z in \mathbb{C}^n such that $\|y\|_2 = \|z\|_2 = 1$, $\|y\|_2 = R_{2,4}\|y\|_4$ and $\|z\|_3 = R_{3,1}\|z\|_1$. By Proposition 1.15, there exists a vector $z_o \in \mathbb{C}^n$ such that $|z_o^*x| \leq \|x\|_3$ and $z_o^*z = \|z\|_3$.

Put $A_o = yz_o$. Then $\frac{\|A_o z\|_2}{\|z\|_1} = \frac{\|yz_o^*z\|_2}{\|z\|_1} = \frac{\|y\|_2\|z\|_3}{\|z\|_1} = \|y\|_2 R_{3,1}$. Hence $\|A_o\|_{1,2} \geq \frac{\|y\|_2}{\|y\|_4} R_{3,1}\|y\|_4 = R_{2,4}R_{3,1}\|y\|_4$. On the other hand, $\frac{\|A_o x\|_4}{\|x\|_3} = \frac{\|yz_o^*x\|_4}{\|x\|_3} = \frac{\|y\|_4|z_o^*x|}{\|x\|_3} \leq \|y\|_4$. Thus $\|A_o\|_{3,4} \leq \|y\|_4$. Hence $\frac{\|A_o\|_{1,2}}{\|A_o\|_{3,4}} \geq \frac{R_{2,4}R_{3,1}\|y\|_4}{\|y\|_4} = R_{2,4}R_{3,1}$. \square

Corollary 2.4 (i) $\|\cdot\|_{1,2} \leq \|\cdot\|_{3,2}$ if and only if $\|\cdot\|_1 \geq \|\cdot\|_3$,

(ii) $\|\cdot\|_{1,2} \leq \|\cdot\|_{1,4}$ if and only if $\|\cdot\|_2 \leq \|\cdot\|_4$.

Proof. (i) $\|\cdot\|_{1,2} \leq \|\cdot\|_{3,2}$ if and only if $\max\{\frac{\|A\|_{1,2}}{\|A\|_{3,2}} : A \neq 0\} = R_{2,2}R_{3,1} \leq 1$ and this if and only if $R_{3,1} \leq 1$ or equivalently $\|\cdot\|_3 \leq \|\cdot\|_1$. The proof of (ii) is similar. \square

The following corollary completely answers to Question (iv):

Corollary 2.5 $\|\cdot\|_{1,2} = \|\cdot\|_{3,4}$ if and only if there exists $\gamma > 0$ such that $\|\cdot\|_1 = \gamma\|\cdot\|_3$ and $\|\cdot\|_2 = \gamma\|\cdot\|_4$.

Proof. If $\|A\|_{1,2} = \|A\|_{3,4}$, then $R_{4,2}R_{1,3} = \max\{\frac{\|A\|_{3,4}}{\|A\|_{1,2}} : A \neq 0\} = 1 = \max\{\frac{\|A\|_{1,2}}{\|A\|_{3,4}} : A \neq 0\} = R_{2,4}R_{3,1}$. Hence $\max\{\frac{\|x\|_2}{\|x\|_4} : x \neq 0\} = R_{2,4} = \frac{1}{R_{3,1}} = \min\{\frac{\|x\|_1}{\|x\|_3} : x \neq 0\} \leq \max\{\frac{\|x\|_1}{\|x\|_3} : x \neq 0\} = R_{1,3} = \frac{1}{R_{4,2}} = \min\{\frac{\|x\|_2}{\|x\|_4} : x \neq 0\}$. Thus there exists a number γ such that $\frac{\|x\|_2}{\|x\|_4} = \gamma = \frac{\|x\|_1}{\|x\|_3}$. \square

Remark 2.6 It is known that each induced norm $\|\cdot\|$ is minimal in the sense that for any matrix norm $\|\cdot\|$, the inequality $\|\cdot\| \leq \|\cdot\|_{1,1}$ implies that $\|\cdot\| = \|\cdot\|_{1,1}$. But this is not true for g-ind norms in general. For instance, put $\|\cdot\|_\alpha = \ell_\infty(\cdot)$, $\|\cdot\|_\beta = 2\ell_2(\cdot)$ and $\|\cdot\|_\gamma = \ell_2(\cdot)$. Then $\|\cdot\|_{\gamma,\beta} \leq \|\cdot\|_{\alpha,\beta}$ but $\|\cdot\|_{\gamma,\beta} \neq \|\cdot\|_{\alpha,\beta}$.

The following theorem is one of our main theorems and provide a complete solution for Problem (v):

Theorem 2.7 *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on \mathbb{C}^n . Then $\|\cdot\|_{1,2}$ is an algebra norm on M_n if and only if $\|\cdot\|_1 \leq \|\cdot\|_2$.*

Proof. For each A and B in M_n we have

$$\|ABx\|_2 \leq \|A\|_{1,2}\|Bx\|_1 \leq \|A\|_{1,2}\|Bx\|_2 \leq \|A\|_{1,2}\|B\|_{1,2}\|x\|_1.$$

Hence $\|AB\|_{1,2} \leq \|A\|_{1,2}\|B\|_{1,2}$.

Conversely, let $\|\cdot\|_{1,2}$ be an algebra norm. Then for each $A, B \in M_n$ we have $\|AB\|_2 \leq \|A\|_{1,2}\|B\|_{1,2}\|x\|_1$. Let B be an arbitrary member of M_n . For $Bx \neq 0$, take M to be the linear span of $\{Bx\}$ and define $f : (M, \|\cdot\|_1) \rightarrow \mathbb{C}$ by $f(cBx) = \frac{c\|Bx\|_1}{\|Bx\|_2}$. By the Hahn-Banach Theorem, there is an $F : (\mathbb{C}^n, \|\cdot\|_1) \rightarrow \mathbb{C}$ with $F|_M = f$ and $\|F\| = \|f\| = \max\{|f(cBx)| : \|cBx\|_1 = 1\} = \max\{\frac{|c|\|Bx\|_1}{\|Bx\|_2} : |c|\|Bx\|_1 = 1\} = \frac{1}{\|Bx\|_2}$. Define $A : (\mathbb{C}^n, \|\cdot\|_1) \rightarrow (\mathbb{C}^n, \|\cdot\|_2)$ by $Ay = F(y)Bx$. Then $\|A\|_{1,2} = \max\{\|Ay\|_2 : \|y\|_1 = 1\} = \max\{|F(y)|\|Bx\|_2 : \|y\|_1 = 1\} = 1$, and $\|ABx\|_2 = |F(Bx)|\|Bx\|_2 = |f(Bx)|\|Bx\|_2 = \frac{\|Bx\|_1}{\|Bx\|_2}\|Bx\|_2 = \|Bx\|_1$. Thus for all B ,

$$\|Bx\|_1 = \|ABx\|_2 \leq \|A\|_{1,2}\|B\|_{1,2}\|x\|_1 = \|B\|_{1,2}\|x\|_1,$$

or

$$\|Bx\|_1 \leq \|B\|_{1,2}\|x\|_1.$$

Now take N to be the linear span of $\{x\}$ and define $g : (N, \|\cdot\|_1) \rightarrow \mathbb{C}$ by $g(cx) = \frac{c\|x\|_1}{\|x\|_2}$. By the Hahn-Banach Theorem, there is a $G : (\mathbb{C}^n, \|\cdot\|_1) \rightarrow \mathbb{C}$ with $G|_N = g$ and $\|G\| = \|g\| = \max\{|g(cx)| : \|cx\|_1 = 1\} = \max\{\frac{|c|\|x\|_1}{\|x\|_2} : |c|\|x\|_1 = 1\} = \frac{1}{\|x\|_2}$. Define $B : (\mathbb{C}^n, \|\cdot\|_1) \rightarrow (\mathbb{C}^n, \|\cdot\|_2)$ by $By = G(y)x$. Then $\|B\|_{1,2} = \max\{\|By\|_2 : \|y\|_1 = 1\} = \max\{|G(y)|\|x\|_2 : \|y\|_1 = 1\} = \|x\|_2\|G\| = 1$, and $\|Bx\|_1 = |G(x)|\|x\|_1 = |g(x)|\|x\|_1 = \frac{\|x\|_1}{\|x\|_2}\|x\|_1 = \frac{\|x\|_1^2}{\|x\|_2}$.

So

$$\frac{\|x\|_1^2}{\|x\|_2} = \|Bx\|_1 \leq \|B\|_{1,2}\|x\|_1 = \|x\|_1.$$

Thus $\|\cdot\|_1 \leq \|\cdot\|_2$. \square

Proposition 2.8 Suppose that $\|\cdot\|_{1,2}$ is a g -ind norm and $\lambda > 0$. Then the scaled norm $\lambda\|\cdot\|_{1,2}$ is a g -ind algebra norm if and only if $\lambda \geq R_{1,2}$.

Proof. Evidently, $\lambda\|\cdot\|_{1,2} = \|\cdot\|_{\|\cdot\|_1, \lambda\|\cdot\|_2}$. If $\|\cdot\|_{3,4} = \lambda\|\cdot\|_{1,2} = \|\cdot\|_{\|\cdot\|_1, \lambda\|\cdot\|_2}$ then Corollary 2.5 implies that there exists $\alpha > 0$ such that $\|\cdot\|_3 = \alpha\|\cdot\|_1$ and $\|\cdot\|_4 = \alpha\lambda\|\cdot\|_2$. Now Theorem 2.7 follows that $\lambda\|\cdot\|_{1,2} = \|\cdot\|_{3,4}$ is an algebra norm if and only if $\alpha\|\cdot\|_1 \leq \alpha\lambda\|\cdot\|_2$ or equivalently $R_{1,2} \leq \lambda$. \square

Proposition 2.9 Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on \mathbb{C}^n and $0 \neq \alpha, \beta \in \mathbb{C}$. Define $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on \mathbb{C}^n by $\|x\|_\alpha = \|\alpha x\|_1$ and $\|x\|_\beta = \|\beta x\|_2$, respectively. Then $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two norms on \mathbb{C}^n and $\|\cdot\|_{\alpha,\beta} = |\frac{\beta}{\alpha}|\|\cdot\|_{1,2}$.

Proof. We have $\|A\|_{\alpha,\beta} = \max\{\|Ax\|_\beta : \|x\|_\alpha = 1\} = \max\{\|\beta Ax\|_2 : \|\alpha x\|_1 = 1\} = |\frac{\beta}{\alpha}| \max\{\|Ay\|_2 : \|y\|_1 = 1\} = |\frac{\beta}{\alpha}|\|A\|_{1,2}$. \square

The preceding proposition leads us to give the following definition:

Definition 2.10 Let $(\|\cdot\|_1, \|\cdot\|_2)$ and $(\|\cdot\|_3, \|\cdot\|_4)$ be two pairs of norms on \mathbb{C}^n . We say that $(\|\cdot\|_1, \|\cdot\|_2)$ is generalized induced congruent (gi-congruent) to $(\|\cdot\|_3, \|\cdot\|_4)$ and we write $(\|\cdot\|_1, \|\cdot\|_2) \equiv_{gi} (\|\cdot\|_3, \|\cdot\|_4)$ if $\|\cdot\|_{1,2} = \gamma\|\cdot\|_{3,4}$ for some $0 < \gamma \in \mathbb{R}$.

Clearly \equiv_{gi} is an equivalence relation. We denote by $[(\|\cdot\|_1, \|\cdot\|_2)]_{gi}$ the equivalence class of $(\|\cdot\|_1, \|\cdot\|_2)$. Proposition 2.9 shows that for each $0 < \alpha, \beta \in \mathbb{R}$ we have $(\alpha\|\cdot\|_1, \beta\|\cdot\|_2) \equiv_{gi} (\|\cdot\|_1, \|\cdot\|_2)$. Indeed, we have the following result:

Theorem 2.11 For each pair $(\|\cdot\|_1, \|\cdot\|_2)$ of norms on \mathbb{C}^n we have $[(\|\cdot\|_1, \|\cdot\|_2)]_{gi} = \{(\alpha\|\cdot\|_1, \beta\|\cdot\|_2) : 0 < \alpha, \beta \in \mathbb{R}\}$.

We can extend the above method to find some other norms on M_n which are not necessarily gi-congruent to a given pair $(\|\cdot\|_1, \|\cdot\|_2)$:

Proposition 2.12 Let $(\|\cdot\|_1, \|\cdot\|_2)$ be a pair of norms on \mathbb{C}^n and $K, L \in M_n$ be two invertible matrices. Define $\|\cdot\|_K$ and $\|\cdot\|_L$ on \mathbb{C}^n by $\|x\|_K = \|Kx\|_1$ and $\|x\|_L = \|Lx\|_2$. Then $\|\cdot\|_K$ and $\|\cdot\|_L$ are norms on \mathbb{C}^n and $\|A\|_{K,L} = \|LAK^{-1}\|_{1,2}$.

Proof. Clear and see also Lemma 3.1 of [4].□

Remark 2.13 Note that the case $K = \alpha I$ and $L = \beta I$ gives Proposition 2.9.

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